Absence of supersensitivity to small input signals in generalized on–off systems

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It has recently been shown that nonlinear skew product dynamical systems with invariant subspaces which are capable of displaying on–off intermittency can show supersensitivity to small input signals.

Here we show that this supersensitivity is absent for more general dynamical systems with non–skew product structure, capable of displaying a generalized form of on–off intermittency, and is therefore in this sense fragile. This absence of supersensitivity is of importance in view of the fact that dynamical systems are generically expected to be of non–skew product nature.

Many dynamical systems of interest possess symmetries or constraints which force the presence of invariant subspaces. A great deal of effort has recently gone into the study of such systems (see e.g. [1–3]). A sub–class of these models, namely those with skew product structure (and normal parameters[1]), have been shown to be capable of producing a number of novel modes of behavior, including on–off intermittency [2] and bubbling [3].

Recently, Zhou and Lai [4] have shown that systems of this type can display supersensitivity, in the sense that small constant or time varying inputs to the system can induce extremely large responses. The authors further claim that with an additional odd symmetry condition, this supersensitivity is robust to addition of noise. Such supersensitivity could be of importance in many fields, including the study of synchronization of coupled chaotic systems [5] and the design of sensor devices [6].

The results on on–off intermittent systems reported by these authors can all be described within the framework of skew product systems. Generically, however, one would expect typical dynamical systems to have non–skew product structure (with non–normal parameters). Systems of this type have recently been studied and have been found to be capable of displaying a number of additional novel dynamical modes of behavior, absent in skew product systems, including a generalization of on–off intermittency, referred to as *in–out intermittency* [7].

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[1]Parameters that leave the dynamics on the invariant manifold unchanged are called normal, otherwise they are referred to as non–normal.

The easiest way to characterise in–out intermittency is by contrasting it with on–off intermittency. Briefly, let M_I be the invariant subspace and A the attractor which exhibits either on–off or in–out intermittency. If the intersection $A_0 = A \cap M_I$ is a minimal attractor, then we have on–off intermittency, whereas if A_0 is not a minimal attractor, then we have in–out intermittency. In the latter case there can be different invariant sets in A_0 associated with attraction and repulsion transverse to A_0, hence the name in–out. Another crucial difference between the two is that, as opposed to on–off intermittency, in the case of in–out intermittency the minimal attractors in the invariant subspaces do not necessarily need to be chaotic and hence the trajectories can (and often do) shadow a periodic orbit in the 'out' phases [7].

Our aim here is to find out whether this type of supersensitivity, observed in on–off intermittent systems, persists in more general non–skew product systems which are capable of displaying in–out intermittent behaviour.

A simple class of maps that can model both on–off and in–out types of intermittency is given by

$$x_{n+1} = F(x_n, y_n, \mathbf{a}),\ \ y_{n+1} = G(x_n, y_n, \mathbf{a}), \tag{1}$$

where $G(x_n, 0, \mathbf{a}) = 0$, the variables x_n and y_n represent the dynamics within the invariant submanifold ($y = 0$) and the transverse distance to it respectively and $\mathbf{a} \in \mathcal{R}^{\mathbf{m}}$ are the control parameters of the system. A special subset of these systems, with skew product form over the dynamics in x, can be written as

$$x_{n+1} = F(x_n, \mathbf{a}),\ \ y_{n+1} = G(x_n, y_n, \mathbf{a}). \tag{2}$$

By considering a skew product system of type (2), Zhou and Lai [4] modelled the motion near the invariant submanifold $y = 0$, using a Fokker–Planck equation. In this way they were able to predict that the sensitivity S of the map in the neighbourhood of a blowout bifurcation, leading to on–off intermittency, is given by

$$S = \frac{\langle y \rangle}{p} = \frac{\tau}{p \ln(\tau/p)}, \tag{3}$$

where $\langle y \rangle$ is the average of the transverse variable y, p is the input signal and τ is the threshold below which y goes through a laminar phase. They were able to confirm this prediction numerically.

To study whether non–skew product (in–out intermittent) systems can also display supersensitivity, we considered a particular example of the map (1) in the form

$$\begin{aligned} x_{n+1} &= r x_n (1 - x_n) + s x_n y_n^2, \\ y_{n+1} &= \nu e^{b x_n} y_n + a y_n^3, \end{aligned} \tag{4}$$

where $r \in [0,4]$ and $(s, \nu, a, b) \in \mathcal{R}$ are the control parameters. Note that for $s = 0$, the map (4) has the skew product form (2) and for fixed r, the parameters s, a, b or ν are normal. Thus depending upon the choice of its parameters, this map is capable of displaying both on–off and in–out types of intermittency, Note also that this map possesses the odd symmetry condition $G(-y) = -G(y)$ that was found in [4] to be required for the robustness of supersensitivity with respect to noise. Also the transverse Lyapunov exponent λ_T for this map can be readily calculated to be

$$\lambda_T = \ln \nu + b\langle x \rangle_r, \tag{5}$$

where $\langle x \rangle_r$ is the average of the variable x_n for an initial condition on the invariant submanifold $y = 0$.

To study the effects of an input signal on in–out systems, we considered a variant of this map given by

$$\begin{aligned} x_{n+1} &= r x_n(1 - x_n) + s x_n y_n^2, \\ y_{n+1} &= \nu e^{b x_n} y_n + a y_n^3 + p, \end{aligned} \tag{6}$$

where the real parameter p models the effects of a small input signal.

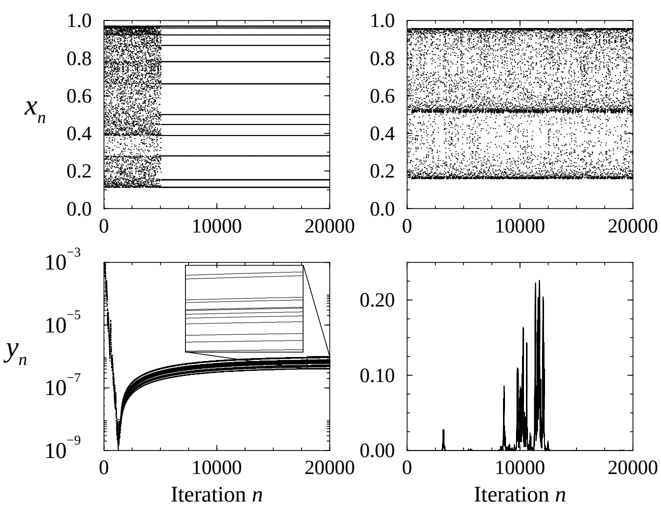

FIG. 1. Comparative study of bursting behaviour in in–out (left panels) and on–off (right panels) regimes, for an input signal $p = 10^{-10}$. Note how in–out dynamics (lower left panel) is insensitive to the input signal, in comparison with on–off (lower right panel). The upper left panel and the inset on the lower left panel also demonstrate clearly the presence of the period 12 attractor in the invariant submanifold. The parameters values are $r = 3.8800045$, $\nu = 1.82$, $b = -1.020625$, $a = -1$ and $s = -0.3$ for the in–out case and $r = 3.82786$, $\nu = 1.82$, $b = -1.006$, $a = -1$ and $s = 0$ for the on–off case.

To begin with, we made a comparative numerical study of the sensitivity of in–out and on–off systems to input signals p, using (6). Fig. 1 shows a comparative study of the bursting behaviour in the two cases close to, but below, their blowout points. As can be seen from the comparison of the lower panels, there is very little bursting in the in–out case.

To further demonstrate this relative insensitivity in the in–out case, we made a study of the sensitivity S of the systems close to their blowout points, as a function of the input signal p. This is shown in Fig. 2, which again demonstrates a distinct absence of supersensitivity for the in–out case, specially for the lower input signal levels. Furthermore, it shows a saturation in sensitivity in the in–out case for input signals $p \lesssim 10^{-7}$.

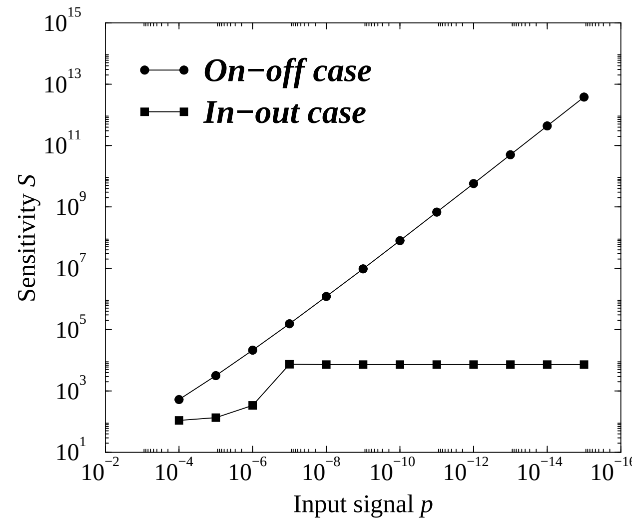

FIG. 2. Dependence of the sensitivity S on the input signal p for in–out and on–off cases. Note the relative insensitivity of the in–out to the input signal and the saturation in sensitivity in this case. The parameter values are as in Fig. 1.

These results indicate a clear lack of supersensitivity in the in–out case to small constant input signals.

To understand this qualitative difference between the on–off and in–out cases, we briefly recall a number of differences between the two cases, relevant to our discussion here. In the case of on–off, the attraction and the ejection of the orbits near the invariant manifold are brought about by a single chaotic attractor in M_I. Thus for the values of the control parameter close to but below the blowout point, the chaotic attractor in M_I becomes transversally weakly attracting, but there can be repelling orbits within this attractor that are transversally unstable, leading to bubbling and allowing the orbit to access the lower and upper boundaries frequently. The system thus becomes sensitive to small inputs, producing large bursts and hence supersensitivity.

For the in–out case, on the other hand, the 'in' and 'out' phases are governed by two separate invariant sets in M_I: a chaotic saddle and a periodic attractor respectively. Thus for the values of control parameter (b in our case) above the blowout value, the chaotic saddle in the invariant submanifold is transversally attracting whereas the periodic attractor in M_I is transversally unstable with a positive transverse Lyapunov exponent λ_T. As a result, an orbit drawn towards the invariant submanifold by the chaotic saddle is thus ejected by the

transversally unstable periodic attractor, leading to in–out intermittency. On the other hand, for the values of control parameter b just below the blowout value, the unique periodic attractor in the invariant submanifold becomes transversally stable (with $\lambda_T < 0$), while the chaotic saddle still remains transversally attracting. As a result orbits drawn towards the invariant submanifold by the chaotic saddle get attracted to the periodic orbit there (see Fig. 3 for a schematic depiction).

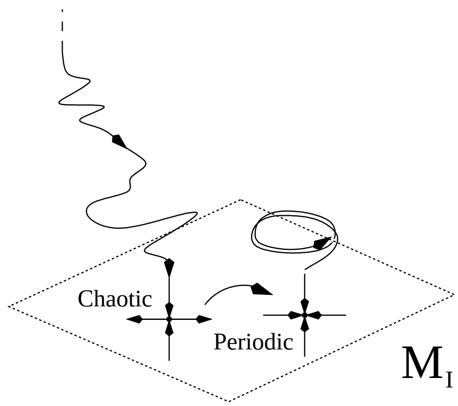

FIG. 3. Schematic diagram showing the main dynamical features of the in–out process near the blowout point with $\lambda_T < 0$. This picture does not qualitatively change in presence of a input signal p, even though the period 12 attractor and the chaotic saddle are slightly shifted from the previous M_I (represented by the dotted line).

We shall now show that the presence of a small input signal p leaves the above dynamical picture essentially unaltered, apart from displacing the location of the periodic attractor off the previous invariant submanifold. There are two ways to see this. Firstly, for small values of the input signal $p \ll \mathcal{O}(1)$, the periodic orbit is expected to persist by continuity. We numerically confirmed that the period 12 orbit involved in the in–out intermittency studied here (see [7] for details) does indeed survive for small values of p, albeit shifted slightly off the invariant submanifold M_I (see the left panels of Fig. 1).

Alternatively, we can estimate the transverse location of the displaced periodic orbit. To do this we recall that we are interested in small displacements from M_I, which implies that as a first approximation we may ignore higher order dependence on y. We therefore approximate the map (6) by

$$\begin{aligned} x_{n+1} &= rx_n(1-x_n) + sx_n y_n^2, \\ y_{n+1} &\approx \nu e^{bx_n} y_n + p, \end{aligned} \tag{7}$$

where the second order term in y has been kept in the x map in order to ensure the essential overall non–skew product structure of the system.

The period 12 attractor involved in this case has x_n values satisfying $x_{n+12} = x_n$ and y_n values given by

$$\begin{aligned} y_0 &= p, \\ y_1 &= \left(\nu e^{bx_0} + 1\right) p, \\ y_2 &= \left(\nu^2 e^{b(x_0+x_1)} + \nu e^{bx_1} + 1\right) p, \\ &\vdots \\ y_{12} &= \left(\nu^{12} e^{b(x_0+x_1+\ldots+x_{11})} + \ldots + \nu e^{bx_{11}} + 1\right) p, \\ y_{13} &= \Big(\nu^{13} e^{b(2x_0+x_1+\ldots+x_{11})} \\ &\quad + \nu^{12} e^{b(x_0+x_1+\ldots+x_{11})} + \ldots + \nu e^{bx_0} + 1\Big) p, \\ &\vdots \\ y_n &= \sum_{j=0}^{n} \left(\nu^j e^{b\left[\sum_{i=1}^{12} \left\lfloor \frac{j+11-(i \bmod 12)}{12} \right\rfloor x_{(i+n-1) \bmod 12}\right]} \right) p, \end{aligned}$$

where $\lfloor\ \rfloor$ denotes the integer part.

The above expressions for y_n change periodically (with period 12), depending on the initial x. The asymptotic average value of y can then be approximated by

$$\begin{aligned} \lim_{n\to\infty} \langle y \rangle &\approx \sum_{j=0}^{n} \left(\nu^j e^{b\left[\sum_{i=0}^{11} \frac{j}{12} x_i\right]} \right) p, \\ &= \left(1 - \nu e^{\frac{b}{12}\sum_{i=0}^{11} x_i} \right)^{-1} p \\ &= \frac{p}{1-\nu e^{\lambda_T - \ln \nu}} = \frac{p}{1-e^{\lambda_T}}, \end{aligned} \tag{8}$$

where we have used (5). Interestingly this enables us to find the sensitivity S as a function of λ_T

$$S = \frac{1}{1-e^{\lambda_T}}, \tag{9}$$

which is independent of p, thus explaining the saturation in sensitivity S observed in the in–out case in Fig. 2, in clear contrast to expression for the on–off sensitivity given by (3) .

It now remains to show that apart from the above shift off the submanifold, the periodic attractor remains essentially intact. To see this, recall that the effect of a non–zero p on x is given by

$$x_{n+1} \sim F^n(x_1, \mathbf{a}) + sx_n \langle y \rangle^2, \tag{10}$$

where $F^n(x_1, \mathbf{a})$ represents the x component of the n^{th} iterate of the map (2). Using (8) this gives

$$|s| x_n \langle y \rangle^2 \sim |s|\, \mathcal{O}(1) \left[\frac{p}{1-e^{\lambda_T}} \right]^2. \tag{11}$$

Now for input signals $p \ll \mathcal{O}(1)$ and for the parameter regimes chosen here, $1 - e^{\lambda_T} \gg \sqrt{p}$, which implies

$$|s| x_n \langle y \rangle^2 \ll p, \tag{12}$$

showing that to this approximation the p-induced variations in x are extremely small (relative to p), hence providing a good indication that the periodic attractor remains essentially intact.

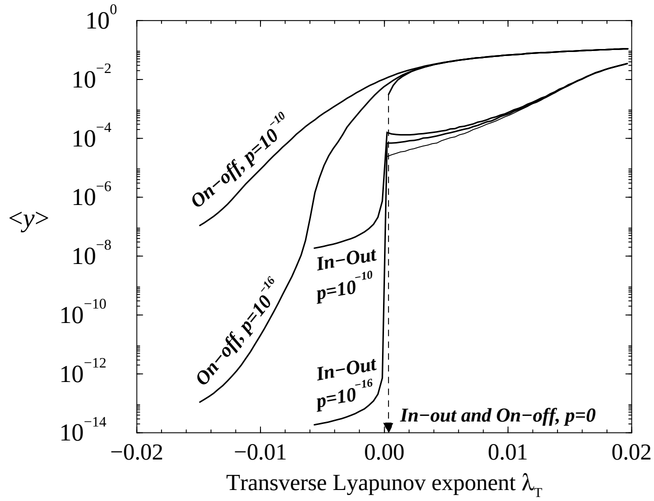

FIG. 4. Dependence of the average blowout variable $\langle y \rangle$ as a function of λ_T, for fixed input signals. The parameters values are as in Fig. 1.

The above arguments and results demonstrate the qualitative differences between the responses of the on–off and the in–out dynamics to small input signals. In particular, the survival of the periodic orbit in the latter case acts to trap the incoming orbits and therefore blocks the possibility of supersensitivity in this case. We expect this picture to be common and thus supersensitivity to be absent in the generic non–skew product (in–out) settings.

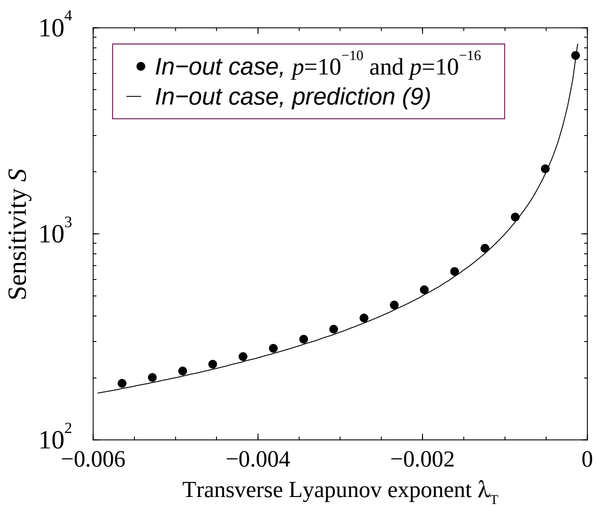

FIG. 5. Dependence of the sensitivity S for a small input signal p as a function of λ_T, together with the predicted scaling (9). The parameter values are as in Fig. 1.

To further substantiate this finding, we calculated the average blowout variable $\langle y \rangle$ in system (6) for fixed input signals, as a function of λ_T. The results are summarised in Fig. 4, which show that for $\lambda_T > 0$, both cases are relatively insensitive to input signals, whereas for $\lambda_T < 0$, the on–off case is much more sensitive to input signals than the in–out case, with the latter dependence in very good agreement with our prediction (8).

Finally we calculated the dependence of the sensitivity S for the in–out case as a function of λ_T, with different input signals, as a function of λ_T. The results are shown in Fig. 5, together with our predicted expression (9), which show excellent agreement.

To summarise, we have argued that the supersensitivity found in [4] for the case of skew product systems with on–off intermittency is absent in the more general setting of non–skew product systems, capable of displaying in–out intermittency. We have substantiated this claim both analytically and through extensive numerical simulations. We have also checked that the absence of supersensitivity in the in–out case remains robust to changes in both the input signal (of the form $p \sin(2\pi x)$) as well as to unbiased noise in the transverse direction (of the order of the input signal).

The absence of supersensitivity for systems displaying in–out intermittency is important, particularly given that dynamical systems are generically expected to be of non–skew product type.

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